Calculation of an Effective Slip in a Settling Suspension at a Vertical Wall

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Geigenmüller and Mazur (1988) demonstrated theoretically that an intrinsic convection develops in a homogeneous suspension settling in a cylindrical container with vertical walls. In large containers, the influence of the walls on the macroscopic suspension motion is described by the effective wall slip velocity. It is shown that the slip velocity in a dilute suspension of spherical particles can be expressed in terms of the induced force monopole and stresslet on a particle sedimenting in the presence of a vertical plane wall. The slip velocity calculated for a dilute hard-sphere suspension is $4.395\phi \nu_s$, where $\phi$ is the particle volume fraction and $\nu_s$ is the Stokes settling velocity.

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Key words: Sedimentation; intrinsic convection; creeping flow.

1. Introduction

The dependence of sedimentation on the shape of a container in statistically uniform suspensions has been explicitly demonstrated (Beenakker and Mazur, 1985; Geigenmüller and Mazur, 1988). A global recirculation of the suspension, called intrinsic convection, is superimposed on the settling motion of the particles. As has been theoretically shown by Geigenmüller and Mazur (1988), particles near the center of a container with vertical walls sediment more rapidly than those closer to the sides. Nozières (1987) proposed a description of this effect in terms of phenomenological macroscopic equations supplemented by effective boundary conditions. Felderhof (1988) developed a statistical theory with macroscopic equations derived from an averaging procedure. The dependence of average particle and fluid motion on the container shape...
results from the long-range character of hydrodynamic interactions; it has, as well, a close analogy in dielectric systems (Peterson and Fixman, 1963; Felderhof, 1976) where macroscopic electrostatic fields depend strongly on geometry.

A basic mechanism of intrinsic convection was illustrated by the simple suspension model proposed by Bruneau et al. (1996). In this model, a particle is represented by a point force (Stokeslet) and steric effects of the container walls are included by forbidding particle-wall overlap. The average suspension motion, described by the Stokes equations with stick boundary conditions on the container walls, is driven by a distribution of point forces. The point force density vanishes within a particle radius from the wall as a result of the excluded-volume effect. This causes the suspension to move upwards near the walls and downwards near the center of the container. Using this model, the flow pattern in long vertical cylinders was analyzed using a boundary layer approach. The analysis predicts Poiseuille flow in the bulk with an effective slip velocity at the walls. The point-force suspension model captures the essential mechanism of intrinsic convection but uses a very approximate representation of the particles.

In this paper we present a rigorous analysis for a dilute suspension of hard spheres without using the simplified particle model. We show that the effective wall-slip velocity can be expressed in terms of the force monopole and the stresslet induced on particles settling in the presence of a plane vertical wall. Using the stresslet calculated from the solution of the Stokes equations in bipolar coordinates, highly accurate evaluation of slip velocity is possible for a suspension with particle-wall correlations resulting from the excluded volume next to the wall.

The hard-sphere system is described in Section 2. Section 3 focuses on the boundary layer analysis. Explicit results for the effective slip velocity are given in Section 4. A discussion and conclusions are presented in Section 5. Some details concerning the solution of the Stokes equations for a particle moving along a plane wall are given in the Appendix.

2. Model System

We consider a dilute suspension of $N$ identical spherical rigid particles of radius $a$ settling under the influence of the gravitational field $g = -g\hat{e}_z$ ($\hat{e}_z$ being a unit vector pointing upwards) in a long vertical cylindrical container having a characteristic diameter $D$. The mass densities of the particles and the fluid are $\rho_p$ and $\rho$, respectively. The suspension is statistically uniform far from the walls, with the particle number density $n_0$ and the bulk particle volume fraction $\phi = \frac{4}{3}na^3n_0$. The particle distribution $n(r) = n_0g_w(r)$ is generally nonuniform in the wall region, with $g_w \neq 1$ (while $g_w = 1$ in the bulk). Since particles cannot overlap with the walls, $g_w(r) = \Theta(d(r) - a)[1 + h_\nu(r)]$, where $d(r)$ is the distance from the nearest wall and $\Theta(x)$ is the Heaviside step function. The function $h_\nu$ describes particle-wall correlations other than those directly resulting from the nonoverlap condition. The correlation function $h = 0$ in the simplest case, which is an appropriate assumption for a dilute suspension.

The suspending fluid is incompressible and its viscosity is $\eta$. The Reynolds number based on the container diameter is small. The flow is stationary so that the fluid motion is described by the stationary Stokes equations with the appropriate stick boundary conditions on particle surfaces and cylinder walls.

To describe particle and fluid motion we use the induced-force formalism (Mazur and Bedeaux, 1974), which is convenient for the averaging procedure. In this method, the suspension is described by a velocity field $v(r)$, which is identical to the fluid velocity in the fluid phase and
the solid-body motion in the solid phase. The pressure field $p(r)$ is also extended into the volume $\Omega$ of the fields $v$ and $p$ satisfy the Stokes equations valid everywhere in the volume $\Omega$ of

$$\eta \nabla^2 v - \nabla p = -f', \ \nabla \cdot v = 0,$$

with the stick boundary condition at the container walls:

$$v(r) = 0 \text{ at } r \in \partial \Omega.$$ 

For convenience, the constant force $\rho g$ is incorporated into the pressure. In Eq. (1) the stick boundary conditions at the particle surfaces is represented by the induced

$$f'(r | X_p) = \sum_{i=1}^{N} f_i(r_i'; r),$$

where $X_p = (r_1, \ldots, r_N)$, and $f_i(r_i'; r)$ is the force density induced on the surface of the particle, with $r_i' = r - r_i$ denoting the position with respect to the particle center at density $f_i$ satisfies the conditions:

$$\int dr_i f_i(r_i'; r) = F,$$

$$\int dr_i r_i \times f_i(r_i'; r) = T = 0,$$

where

$$F = \frac{4 \pi a^3}{3} (\rho_p - \rho) g$$

is the gravitational force acting on a particle and the torque $T$ is zero because assumed to rotate freely.

Due to hydrodynamic interactions, $f_i$ is a complicated function of the positions of the particles. In dilute suspensions, the hydrodynamic interactions between particles can be neglected because they do not contribute to the average suspension motion in leading order. However, nonuniformity of the induced force close to the wall produces an $O(\phi)$ contribution.

The macroscopic suspension velocity is identified with the ensemble average of the average Stokes equations resulting from Eq. (1):

$$\eta \nabla^2 \langle v \rangle - \nabla \langle p' \rangle = -\langle f' \rangle - n_0 F,$$

where $\langle p' \rangle = \langle p \rangle + n_0 F$, with $F = |F|$. In the bulk of the container $\langle f' \rangle = n_0 F$. This does not hold in the wall region as a result of nonuniformity of the particle distribution and redistribution of the induced force on particle surfaces.
the solid-body motion in the solid phase. The pressure field \( p(r) \) is also extended into the particles. The fields \( \mathbf{v} \) and \( p \) satisfy the Stokes equations valid everywhere in the volume \( \Omega \) of the container:

\[
\eta \nabla^2 \mathbf{v} - \nabla p = - \mathbf{f}, \quad \nabla \cdot \mathbf{v} = 0,
\]

with the stick boundary condition at the container walls:

\[
\mathbf{v}(r) = 0 \text{ at } r \in \partial \Omega.
\]

For convenience, the constant force \( \rho g \) is incorporated into the pressure. In Eq. (1), the effect of the stick boundary conditions at the particle surfaces is represented by the induced force density:

\[
f(r | X) = \sum_{i=1}^{N} f_i(r_i; r),
\]

where \( X = (r_1, \ldots, r_N) \), and \( f_i(r_i; r) \) is the force density induced on the surface of the \( i \)-th particle, with \( r_i = r - r_i \), denoting the position with respect to the particle center at \( r_i \). The force density \( f \) satisfies the conditions:

\[
\int d^3r_i f_i(r_i; r) = F,
\]

\[
\int d^3r_i \times f_i(r_i; r) = T = 0,
\]

where

\[
F = \frac{4\pi a^3}{3} (\rho_p - \rho) g
\]

is the gravitational force acting on a particle and the torque \( T \) is zero because particles are assumed to rotate freely.

Due to hydrodynamic interactions, \( f \) is a complicated function of the positions of all particles. In dilute suspensions, the hydrodynamic interactions between particles can be neglected because they do not contribute to the average suspension motion in leading order \( O(\phi) \). The particle-wall hydrodynamic interactions, however, cannot be neglected since the resulting nonuniformity of the induced force close to the wall produces an \( O(\phi) \) contribution.

The macroscopic suspension velocity is identified with the ensemble average \( \langle \mathbf{v} \rangle \), which satisfies the average Stokes equations resulting from Eq. (1):

\[
\eta \nabla^2 \langle \mathbf{v} \rangle - \nabla \langle p' \rangle = -\langle f \rangle - \eta_0 F,
\]

where \( \langle p' \rangle = \langle p \rangle + \eta_0 x F \), with \( F = |F| \). In the bulk of the container \( \langle f \rangle = \eta_0 F \). This relationship does not hold in the wall region as a result of nonuniformity of the particle distribution near the walls and redistribution of the induced force on particle surfaces.
3. Boundary Layer Analysis

For a container width $D$ that is much larger than particle radius $a$, the macroscopic suspension flow can be determined from a boundary layer analysis (Bruneau et al., 1996). To find the inner-region solution (close to the wall), we consider a local Cartesian coordinate system with the origin at the wall, the axis $X$ vertical pointing upwards, $Y$ horizontal along the wall, and $Z$ normal to the wall pointing into the suspension. The scale of the new coordinates is based on the particle radius, so that a particle at $Z = 1$ touches the wall. The inner problem corresponds to a suspension bounded by an infinite vertical plane at $Z = 0$, in the leading order in $a/D$. It follows from symmetry that $\langle v \rangle$ has the form:

$$\langle v \rangle = U_s(Z)u_s \delta_x,$$

where $u_s = F/(6\pi \eta a)$ is the Stokes velocity. The function $U_s(Z)$ satisfies a dimensionless equation

$$\frac{\partial^2}{\partial Z^2} U_s - \frac{\partial}{\partial X} P = -\frac{9}{2} \phi (\tilde{f}_s^l + 1),$$

with the boundary condition

$$U_s(0) = 0,$$

where

$$P = \frac{a}{\eta \nu u_s} \langle p' \rangle, \quad \tilde{f}_s^l = \frac{\langle f_s^l \rangle}{\eta F}.$$  

We note that $f_s^{il}$ is negative in our coordinate system and equals -1 in the bulk of the container. The bracketed term in Eq. (9) describes the deficit of the induced force in the wall region.

A general solution of Eqs. (9) and (10) has the form:

$$U_s(Z) = \frac{1}{2} AZ^2 + BZ + Q(0) - Q(Z),$$

$$P(X) = P_0 + AX,$$

where the parameters $A$, $B$, and $P_0$ are determined by matching to the outer solution, and

$$Q(Z) = \frac{9}{2} \phi \int_{Z}^{\infty} d\tilde{Z} \left( \tilde{Z} - Z \right) (\tilde{f}_s^l(\tilde{Z}) + 1).$$

The matching is done in the region $1 << Z << D/a$ where $\tilde{f}_s^l + 1 = 0$ and the outer solution is given by Eq. (8) with
\[ U_i(Z) = \frac{1}{2} AZ^2 + BZ + Q(0). \]

By formally extending Eq. (15) to the wall, we find the normalized wall-slip velocity

\[ U_w = Q(0) = \frac{9}{2} \phi \int_0^\infty dZ Z \tilde{j}_i^s(Z) + 1. \]

It follows that the outer solution has the form of a Poiseuille flow plus the constant slip given by (8) and (16). The pressure gradient is determined by the zero net flow condition.

To simplify Eq (16) we use the expression

\[ \tilde{j}_i^s(Z) = \int d^3R \ G_w(Z_1) \tilde{j}_i^s(R - R_1; Z_i), \]

where \( G_w(Z_1) = g_w(r) \) is the rescaled particle distribution and \( \tilde{f}_i(R - R_1; Z_i) = (a^2/F) f_i^0 \) is the normalized distribution of the force induced on a particle a distance \( Z_i \) from the slip velocity can be expressed as a sum of two terms:

\[ U_w = U_0 + U_{\text{III}}. \]

The first term,

\[ U_0 = \frac{9}{8} \phi \int_{-1}^\infty dZ_1 \ [G_w(Z_1) - 1](1 + Z_1)^3, \]

corresponds to the uniform normalized induced force distribution

\[ \tilde{f}_i^{00}(R - R_1) = \frac{1}{4\pi} \delta(|R - R_1| - 1)\delta_z, \]

whose average is nonuniform near the wall. The second term \( U_{\text{III}} \) that results from the per

\[ \tilde{f}_i^{\text{III}} = \tilde{f}_i' - \tilde{f}_i^{00} \]

is produced by hydrodynamic interactions between particles and the wall. Since part torque-free, \( U_{\text{III}} \) can be expressed in the form:

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\[ U_{w}(Z) = \frac{1}{2} AZ^{2} + BZ + O(0). \]

By formally extending Eq. (15) to the wall, we find the normalized wall-slip velocity

\[ U_{w} = O(0) = \frac{9}{2} \phi \int_{0}^{\infty} dZ Z \tilde{f}_{i}^{1}(Z) + 1]. \]

It follows that the outer solution has the form of a Poiseuille flow plus the constant slip \( S_{2} \) given by (8) and (16). The pressure gradient is determined by the zero net flow condition.

To simplify Eq. (16), we use the expression

\[ \tilde{f}_{i}^{1}(Z) = \int d^{3}R G_{w}(Z_{i}) \tilde{f}_{i}^{0}(R - R_{i}; Z_{i}), \]

where \( G_{w}(Z_{i}) = g_{w}(r_{i}) \) is the rescaled particle distribution and \( \tilde{f}_{i}^{1}(R - R_{i}; Z_{i}) = (a^{3}/F) f_{i}^{0} \) is the normalized distribution of the force induced on a particle a distance \( Z_{i} \) from the wall. The wall-slip velocity can be expressed as a sum of two terms:

\[ U_{w} = U_{0} + U_{H}. \]

The first term,

\[ U_{0} = -\frac{9}{8} \phi \int_{-1}^{\infty} dZ_{i} [G_{w}(Z_{i}) - 1](1 + Z_{i})^{3}, \]

corresponds to the uniform normalized induced force distribution

\[ \tilde{f}_{i}^{0}(R - R_{i}) = \frac{1}{4\pi} \delta(|R - R_{i}| - 1) \delta_{x} \]

whose average is nonuniform near the wall. The second term \( U_{H} \) that results from the perturb

\[ \delta f_{i}^{H} = \tilde{f}_{i} - \tilde{f}_{i}^{0} \]

is produced by hydrodynamic interactions between particles and the wall. Since particle torque-free, \( U_{H} \) can be expressed in the form:

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\[ U_s(Z) = \frac{1}{2}AZ^2 + BZ + Q(0). \]  
(15)

By formally extending Eq. (15) to the wall, we find the normalized wall-slip velocity

\[ U_w = Q(0) = \frac{9}{2} \phi \int_{0}^{\infty} dZ Z\bar{f}_w^i(Z) + 1. \]  
(16)

It follows that the outer solution has the form of a Poiseuille flow plus the constant slip velocity given by (8) and (16). The pressure gradient is determined by the zero net flow condition.

To simplify Eq (16) we use the expression

\[ \bar{f}_w^i(Z) = \int d^3R \, G_w(Z, Z_i) \bar{f}_w^i(R - R_i; Z_i), \]  
(17)

where \( G_w(Z_i) = g_w(r_i) \) is the rescaled particle distribution and \( \bar{f}_w^i(R - R_i; Z_i) = (a^3/F) \, \bar{f}_w^i(R - r_i; r_i) \) is the normalized distribution of the force induced on a particle a distance \( Z_i \) from the wall. The slip velocity can be expressed as a sum of two terms:

\[ U_w = U_b + U_{II}. \]  
(18)

The first term,

\[ U_b = -\frac{9}{8} \phi \int_{-\infty}^{\infty} dZ_i \left[ G_w(Z_i) - 1 \right](1 + Z_i^2), \]  
(19)

corresponds to the uniform normalized induced force distribution

\[ f_{w}^{(0)}(R - R_i) = \frac{1}{4\pi} \delta(|R - R_i| - 1) \delta_x \]  
(20)

whose average is nonuniform near the wall. The second term \( U_{II} \) that results from the perturbation

\[ \delta \bar{f}_{II}^i = \bar{f}_w^i - f_{w}^{(0)} \]  
(21)

is produced by hydrodynamic interactions between particles and the wall. Since particles are torque-free, \( U_{II} \) can be expressed in the form:
\[ U_{\text{III}} = \frac{3}{2} \phi \int_0^\infty dZ_i G_\phi(Z_i) S_{xz}(Z_i), \]  

where

\[ S_{xz}(Z_i) = \frac{1}{2} \int d^3R' [X' f_{1,4}^d(R', Z_i) + Z' f_{1,4}^d(R'; Z_i)] \]  

is the \( xZ \) component of the normalized induced stresslet on a particle a distance \( Z_i \) from the wall. (We note that the effective wall slip in sheared suspensions has also been expressed as an integral of the stresslet induced on a particle (Brunn, 1981).)

4. Calculation of the Slip Velocity

We present the explicit results for

\[ G_\phi(Z_i) = \Theta(Z_i - 1), \]  

i.e., when there are no particle-wall correlations except those resulting from the nonoverlap condition. For such \( G_\phi \), the term \( U_0 \) can be immediately evaluated:

\[ U_0 = 3\phi, \]  

in agreement with Geigenmüller and Mazur (1988). Note that this value differs from the slip velocity (9/4\( \phi \)) found in Bruneau et al. (1996) and Feuillebois et al. (1996), where spheres were modeled by point forces rather than the uniform distribution (Eq. 20).

To evaluate \( U_{\text{III}} \), the stresslet induced on a particle settling freely in the presence of a fixed vertical plane is needed. We have calculated \( S_{xz} \) using the solution in bipolar coordinates by Dean and O'Neil (1963) and O'Neil (1964). Some details of the calculation are described in Appendix A. The induced stresslet is plotted in Fig. 1 as a function of the distance \( Z \) from the wall. The figure shows the accurate result from the bipolar-coordinates solution; the \( O(Z^3) \) term in the expansion of \( S_{xz} \) in inverse powers of \( Z \) (Kim and Karilla, 1991):

\[ S_{xz} = \frac{5}{16} Z^{-2} + \delta S_{xz}, \quad \delta S_{xz} = o(Z^{-2}); \]  

and the difference \( \delta S_{xz} \) between the exact and approximate solutions.

For \( Z < 1.1 \), the difference between the exact solution and the \( O(Z^3) \) term is large. At the touching configuration \( Z = 1 \), the whole contribution to the stresslet results from the forces concentrated in the lubrication region. The vertical component of the total force induced near the contact point is \(-F\) and the particles are torque-free, thus \( S_{xz}(1) = 1 \). The convergence to this value is logarithmic and therefore, very slow (for example, \( S_{xz}(1 + 10^{-4}) = 0.68 \) and \( S_{xz}(1 + 10^{-5}) = 0.77 \)). For \( 1.1 < Z < 3 \) the relative error of the \( O(Z^3) \) approximation is of the order of 10 percent. The \( O(Z^3) \) term is increasingly accurate for larger \( Z \). The correction \( \delta S_{xz} \) changes sign at \( Z \approx 1.2 \).
Fig. 1. Normalized stresslet $S_n$ induced on settling sphere as function of normalized distance $Z$ from an infinite fixed vertical plane. Exact result (solid line); leading term (26) in expansion in the inverse distance from the wall (dashed line); difference between exact and approximate solutions (dash-dotted line).

When Eq. (26) is substituted into Eq. (22) and integrated, the first term contributes $\frac{141}{32}$ to $U_{\text{II}}$. Numerical integration of the second term yields $-0.0114\phi$; the small contribution results from cancellation when integrated over both the positive and negative regions. The complete slip velocity is:

$$U_s = \left[ \frac{141}{32} - 0.0114 \right] \phi = 4.395\phi. \quad (27)$$

because of the slow $O(Z^{-2})$ decay of $S_n$ at large $Z$, matching of the inner and outer solutions at a distance $Z_0$ from the wall results in a $O(Z_0^2)$ error in the slip velocity.

*In Feuillebois et al. (1996), we reported an incorrect value of $U_s$ because of a mistake in the sign of $U_{\text{II}}$. There are also some differences resulting from changes in notation.
5. Discussion and Conclusions

The slip velocity at vertical walls in a dilute sedimenting suspension consists of two contributions. In the monopole contribution the induced force is uniformly distributed over a particle surface. The excluded volume reduces the force density adjacent to the container walls and the force deficit drives the macroscopic flow. The stresslet contribution results from particle-wall hydrodynamic interactions that move the induced forces on a particle towards the wall. The shift of the induced-force deficit away from the wall intensifies the convective suspension motion and the slip velocity increases from $3\phi u_\infty$ to $4.395\phi u_\infty$. The other induced-force multipoles do not affect slip velocity.

The aforementioned values were obtained using simple particle-wall correlations (Eq. (24)) that result entirely from the excluded volume near the wall. Although this is a good assumption for a dilute suspension, various factors (e.g., different procedures used to prepare the initial suspension state) may affect the particle-wall correlation function. Over a long period, different particle-wall correlations may appear during the sedimentation process. While a single sedimenting particle does not change its position with respect to a vertical wall, the accumulation of displacements due to interactions with other particles may produce an $O(1)$ change of correlations even in a dilute suspension. The examples considered in Bruneau et al. (1996) show that the slip velocity is very sensitive to changes in particle distribution near the wall (an inverse convection may appear for some distributions).

Like in equilibrium systems, in sedimenting dense suspensions the simple assumption of Eq. (24) does not hold. Pronounced oscillations of the particle concentration in the wall region are expected. (In sheared suspensions such oscillations have been experimentally observed (Krishnan and Leighton, 1995; and Rampall, 1995).) Assuming equilibrium particle-wall correlations and using a point-force suspension model, it has been estimated (Feuillebois et al., 1996) that the oscillations may result in a substantial reduction in intrinsic convection. A more accurate evaluation of this effect requires a detailed investigation of dynamic particle-wall correlations and many-particle hydrodynamic interactions in the presence of a wall.

APPENDIX A

In order to evaluate the induced stresslet (Eq. 23) for a sphere moving in the presence of a plane wall we use solutions of the Stokes equations in bipolar coordinates (Dean and O'Neill, 1963; O'Neill, 1964; O'Neill, 1967). We consider a particle of radius $a$ moving with velocity $(U, 0, 0)$ and angular velocity $(0, \Omega, 0)$ in the reference frame in which the plane is given by $z = 0$ and the position of the sphere is $(0, 0, d)$. For a freely settling sphere, $U$ and $\Omega$ are determined by the total force exerted on the particle and the condition that the particle is torque-free.

We introduce a cylindrical coordinate system $(\rho, \phi, z)$ with $x = \rho \cos \phi$, $y = \rho \sin \phi$. The pressure $p$ and cylindrical components ($v_\rho$, $v_\phi$, $v_z$) of $v$ have been obtained (Dean and O'Neill, 1963; O'Neill, 1964) in the form we present below, using a slightly different notation:

$$ap = \eta Q_1 \cos \phi, \quad av_\rho = \frac{1}{2}[p Q_1 + c(U_2 + U_0)] \cos \phi$$  \hspace{1cm} (A1)
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$$ap = \eta Q \cos \phi, \quad av_\rho = \frac{1}{2} \rho Q \cos \phi + c(U_z + U_\phi) \cos \phi \quad (A1)$$
\[ a_{v_\phi} = \frac{1}{2} \xi (U_2 - U_0) \sin \phi, \quad a_{v_z} = \frac{1}{2} \xi (Q_1 + 2c w_1) \cos \phi, \]

where

\[ w_1 = (\cosh \xi - \mu)^{1/2} \sin \eta \sum_{n=1}^{\infty} \left[ A_n \sinh \left( n + \frac{1}{2} \right) \xi \right] P_n(\mu), \]

\[ Q_1 = (\cosh \xi - \mu)^{1/2} \sin \eta \sum_{n=1}^{\infty} \left[ B_n \cosh \left( n + \frac{1}{2} \right) \xi + C_n \sinh \left( n + \frac{1}{2} \right) \xi \right] P_n(\mu), \]

\[ U_0 = (\cosh \xi - \mu)^{1/2} \sum_{n=1}^{\infty} \left[ D_n \cosh \left( n + \frac{1}{2} \right) \xi + E_n \sinh \left( n + \frac{1}{2} \right) \xi \right] P_n(\mu), \]

\[ U_2 = (\cosh \xi - \mu)^{1/2} \sin^2 \eta \sum_{n=1}^{\infty} \left[ F_n \cosh \left( n + \frac{1}{2} \right) \xi + G_n \sinh \left( n + \frac{1}{2} \right) \xi \right] P_n(\mu). \]

The bipolar coordinates \((\xi, \eta)\) are defined by the relations:

\[ \rho = \frac{c \sin \eta}{\cosh \xi - \cos \eta}, \quad z = \frac{c \sinh \xi}{\cosh \xi - \cos \eta}, \quad (0 \leq \eta \leq \pi); \]

the constant \(c\) is determined by the equations:

\[ c = a \sinh \alpha, \quad \cosh \alpha = \frac{d}{a}. \]

\[ \mu = \cos \eta; \ P_n(\mu) \text{ is the Legendre polynomial of order } n; \text{ and the prime denotes the differ} \]

with respect to \(\mu\). In bipolar coordinates, \(\xi = 0\) describes the plane and \(\xi = \alpha > 0\) the

Dean and O'Neill (1963) derived an algorithm to determine the expansion \(A_n, \ldots, G_n\) for \(U \neq 0\) and \(\Omega = 0\) and O'Neill (1964) derived one for \(U = 0\) and \(\Omega \neq 0\) solution for a torque-free sphere moving along the wall is a linear combination of these two solutions. Dean and O'Neill (1963) and O'Neill (1964) also derived the expressions for the force on the particle in terms of the coefficients \(A_n, \ldots, G_n\). These expressions were revised by O'Neill (1967).

Figure 2 shows fluid streamlines around a torque-free sphere settling along a wall. Streamlines have been calculated in the bipolar coordinates using the algorithm from E O'Neill (1963) and O'Neill (1964). The streamlines are plotted in the wall referenc
\[ a v_\phi = \frac{1}{2}(U_z - U_0) \sin \phi, \quad a v_z = \frac{1}{2}(\xi Q_1 + 2 \xi w_1) \cos \phi, \] (A2)

where

\[ w_1 = (\cosh \xi - \mu)^{1/2} \sin \eta \sum_{n=1}^{\infty} \left[ A_n \sinh \left( n + \frac{1}{2} \right) \xi \right] P_n(\mu), \] (A3)

\[ Q_1 = (\cosh \xi - \mu)^{1/2} \sin \eta \sum_{n=1}^{\infty} \left[ B_n \cosh \left( n + \frac{1}{2} \right) \xi + C_n \sinh \left( n + \frac{1}{2} \right) \xi \right] P_n(\mu), \] (A4)

\[ U_0 = (\cosh \xi - \mu)^{1/2} \sum_{n=0}^{\infty} \left[ D_n \cosh \left( n + \frac{1}{2} \right) \xi + E_n \sinh \left( n + \frac{1}{2} \right) \xi \right] P_n(\mu), \] (A5)

\[ U_2 = (\cosh \xi - \mu)^{1/2} \sin^2 \eta \sum_{n=0}^{\infty} \left[ F_n \cosh \left( n + \frac{1}{2} \right) \xi + G_n \sinh \left( n + \frac{1}{2} \right) \xi \right] P_n(\mu). \] (A6)

The bipolar coordinates \((\xi, \eta)\) are defined by the relations:

\[ \rho = \frac{c \sin \eta}{\cosh \xi - \cos \eta}, \quad z = \frac{c \sin \xi}{\cosh \xi - \cos \eta}, \quad (0 \leq \eta \leq \pi); \] (A7)

the constant \(c\) is determined by the equations:

\[ c = a \sinh \alpha, \quad \cosh \alpha = \frac{d}{a}, \] (A8)

\(\mu = \cos \eta; P_n(\mu)\) is the Legendre polynomial of order \(n\); and the prime denotes the differentiation with respect to \(\mu\). In bipolar coordinates, \(\xi = 0\) describes the plane and \(\xi = \alpha > 0\) the sphere.

Dean and O'Neil (1963) derived an algorithm to determine the expansion coefficients \(A_n, \ldots, G_n\) for \(U \neq 0\) and \(\Omega = 0\) and O'Neill (1964) derived one for \(U = 0\) and \(\Omega \neq 0\). The solution for a torque-free sphere moving along the wall is a linear combination of these two basic solutions. Dean and O'Neill (1963) and O'Neill (1964) also derived the expressions for the torque and force on the particle in terms of the coefficients \(A_n, \ldots, G_n\). These expressions were later revised by O'Neill (1967).

Figure 2 shows fluid streamlines around a torque-free sphere settling along a wall; these streamlines have been calculated in the bipolar coordinates using the algorithm from Dean and O'Neill (1963) and O'Neill (1964). The streamlines are plotted in the wall reference-frame.
Fig. 2. Fluid streamlines around a freely rotating sphere settling along a vertical wall. The streamlines are shown in the frame of the wall. The gap between the sphere and the wall is \( d - a = 0.1 \ a \).

Note that although the sphere is rotating, the streamlines are symmetric with respect to the plane \( x = 0 \), which follows from the symmetry of the Stokes equations with respect to velocity reversal.

To calculate the stresslet induced on the sphere we generalized the method used by Dean and O’Neill (1963). The stresslet \( S_{\tau \tau} \) has been expressed as:

\[
S_{\tau \tau} = \frac{1}{2} T_y + \delta_{\tau \tau},
\]  

(A9)

where \( T_y \) is the torque exerted on the particle. (The normalizations of \( T_y \) and \( S_{\tau \tau} \) are the same.) To determine \( S_{\tau \tau} \) we use the formula

\[
S_{\tau \tau} = \frac{1}{d^3} \int_{r=a} dS \left[ \frac{xz'}{r} p - \eta \frac{x}{r^2} \left( \frac{\partial v_z}{\partial r} - \frac{v_z}{r} \right) \right],
\]  

(A10)
where integration is over the sphere surface, \( z' = z - d \), and \( r' \) denotes the distance from the center of the sphere (cf. Eq. (3.2.36) in Happel and Brenner (1991)). We have transformed the Eq. (A10) into the bipolar coordinates Eq. (A7) and inserted the Eqs. (A1)-(A4). The result has been simplified using computer algebraic manipulation software (Maple) with the result:

\[
S_n = \frac{2^{12}}{5} \sinh \alpha \sum_{n=1}^{\infty} \left\{ n(n + 1) \left[ 2A_n + \frac{1}{3}(2n + 1)(B_n + C_n) \right] \right\}.
\]

(A11)

Since in our problem \( T_j = 0 \) Eq. (A11) is sufficient to determine the wall slip velocity (Eqs. (18-22)). In a general case, \( T_j \) can be evaluated using Eqs. (9) and (10) from O'Neill (1967).

NOMENCLATURE

- \( a \) - particle radius;
- \( D \) - container width;
- \( \theta_i \) - unit vector pointing upwards;
- \( f^i \) - force density at a point in suspension;
- \( f_i^j \) - force density induced on \( i \)-th particle;
- \( f_i^j \) - normalized force density induced on \( i \)-th particle;
- \( F \) - gravitational force exerted on a particle;
- \( |F| \) - absolute value of \( F \);
- \( g \) - gravitational acceleration;
- \( g_n \) - particle-wall correlation function (particle distribution normalized by \( n_0 \));
- \( G_n \) - particle-wall correlation function as function of normalized distance to wall;
- \( n \) - particle distribution;
- \( n_0 \) - particle number density in bulk of suspension;
- \( p \) - pressure;
- \( P \) - normalized pressure;
- \( r \) - point in suspension;
- \( r_i \) - center of \( i \)-th particle;
- \( r_i' = r - r_i \) - position with respect to center of \( i \)-th particle;
- \( R \) - point in suspension, expressed in normalized coordinate system;
- \( R_i \) - center of \( i \)-th particle in normalized coordinate system;
- \( S_n \) - \( X \) component of normalized stresslet induced on a particle;
- \( U_n \) - vertical component of average suspension velocity normalized by Stokes settling velocity;
- \( U_n \) - normalized slip velocity of suspension on the wall;
- \( U_0 \) - contribution to \( U_n \) from excluded volume effect;
- \( U_{10} \) - contribution to \( U_n \) from hydrodynamic interactions;
- \( \nu \) - velocity of suspension;
- \( \nu_n \) - Stokes settling velocity;
- \( \chi \) - vertical coordinate pointing upwards
- \( \chi \) - vertical coordinate pointing upwards, normalized by sphere radius;
- \( Y \) - horizontal coordinate along the wall, normalized by sphere radius;
- \( Z \) - horizontal coordinate pointing away from the wall, normalized by sphere radius;
- \( \langle \cdot \rangle \) - ensemble average.
\( \eta_f \) - fluid dynamic viscosity;
\( \phi \) - particle volume fraction;
\( \Theta \) - Heaviside step function;
\( \rho \) - mass density of fluid;
\( \rho_p \) - mass density of particles.

REFERENCES


