Three-dimensional intrinsic convection in dilute and dense dispersions of settling spheres

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The three-dimensional intrinsic convection in a monodisperse dispersion of spheres settling in a vertical container of arbitrary cross section is calculated using the simple model of point forces with excluded volume near the walls, proposed by Bruneau et al. [Phys. Fluids 8, 2236 (1996)]. An exact solution of the model equations for a container with rectangular cross-section shows that corners have no significant influence on the convection. A dense suspension is modeled by assuming an equilibrium particle distribution in the near wall region. It is predicted that as a result of near wall ordering, the intrinsic convection decreases with increasing particle volume fraction. © 1998 American Institute of Physics. [S1070-6631(98)01001-0]

I. INTRODUCTION

It has been shown theoretically1–4 that the sedimentation velocity of solid spherical particles in a viscous fluid depends on the shape of the container. More precisely, the particle velocity with respect to the fluid is superimposed on a container-dependent global motion of the suspension. This motion is called “intrinsic convection” because it occurs without particle concentration gradient.

In the simple model proposed by Bruneau et al.,5 spherical particles are replaced by point forces (Stokelets) located at their centers. The condition that particles do not overlap with walls is enforced by keeping a layer free of Stokelets adjacent to each wall. The intrinsic convection results from this key condition: the weight of the suspension is smaller near the walls, driving the convective motion.

Assuming that the radius of spheres is much smaller than the characteristic dimension of the container cross-section, the governing equations can be simplified by a boundary layer analysis. As it was shown in Ref. 5, the inner flow has a form of a Poiseuille flow with a slip velocity at the walls. The value of the slip is very sensitive to the near-wall particle distribution.

The simple point-force model is used here to study in more detail the role of the container geometry and of the near-wall particle distribution in a dense suspension.

In Sec. II we consider a general vertical container and derive the expression for the suspension slip velocity at the wall as a function of a near-wall particle distribution. This formulation is then applied, in Sec. III, to the three dimensional intrinsic convection in a vertical cylinder with a rectangular cross-section, a container appropriate for experiments. The boundary layer approach ignores corners of the container; however, it is shown by an exact solution of the model equations that for the rectangular cylinder the corners have no significant influence on the intrinsic convection.

To estimate the effect of the particle-wall correlations on the intrinsic convection in dense suspensions, the equilibrium near-wall particle distribution is assumed. This distribution is calculated in Sec. IV, using the Percus-Yevick approximation. The result is used, in Sec. V, to estimate the dependence of the wall slip velocity on the suspension concentration.

Finally, the conclusion and discussion are presented in Sec. VI.

II. GENERAL EXPRESSION FOR THE SLIP VELOCITY

We consider a uniform suspension of spherical particles sedimenting in a tall cylinder with vertical side walls. As in Ref. 5, the particles are represented by point forces (Stokelets) located at their centers. The distribution of Stokelets is described by the function \( f(\xi) \), where \( \xi \) is the minimum distance between the current point and the nearest wall. The Stokelet distribution in the point force model is equivalent to the local particle number density. We assume that \( f(\xi) \) does not depend on the position along the wall. As a result of the steric effect of the walls

\[
\xi = 0 \quad \text{for} \quad \xi < a, \\
\xi = 1 \quad \text{for} \quad \xi > \xi_0.
\]

where \( a \) is the radius of the spheres. In Ref. 5 it was assumed that the distribution of Stokelets is uniform except for the depleted region (1). In the present paper we adopt a weaker assumption that the distribution becomes uniform at some distance \( \xi_0 > a \), allowing for nontrivial particle-wall correlations. Since the correlations have their origin in particle-wall interactions we assume that \( \xi_0 = O(a) \). We use the normalization such that

\[
f(\xi) = 1 \quad \text{for} \quad \xi > \xi_0.
\]
\[ \mu \Delta w - \frac{dp}{dz} = \rho_f g + (\rho_p - \rho_f) g e f(\xi), \]  \hspace{1cm} (3) \\
\int_S w dS = 0, \hspace{1cm} \hspace{1cm} (4) \\
w = 0 \text{ on walls}, \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} (5)

where \(\mu\) is the dynamic fluid viscosity, \(p\) the pressure, \(\rho_p\) and \(\rho_f\) are the particle and fluid densities, \(c\) is the particle volume fraction, \(g\) is gravitational acceleration, \(z\) is the vertical coordinate, and \(S\) is the cross section of the cylinder.

Let \(b\) be a characteristic container dimension and let us assume that \(\epsilon = \frac{a}{b} \ll 1\).

By considering a perturbation solution of (3) in terms of \(\epsilon\), one can show that the outer solution (i.e., outside the wall region) is a sum of a constant velocity \(w_\ast\) and a Poiseuille flow induced by the constant reduced pressure gradient \(A\) defined by

\[ A = \frac{dp}{dz} + \rho_f g + (\rho_p - \rho_f) g e c. \]  \hspace{1cm} (6)

The pressure gradient and the velocity \(w_\ast\) are related by the zero net flux condition (4) and \(w_\ast\) is equal to the effective fluid slip velocity at the walls which results from matching of the inner and outer boundary-layer solutions.

The inner solution of (3) has the following form:

\[ w = \frac{1}{2\mu} A \xi^2 + B \xi + Q(0) - Q(\xi), \]  \hspace{1cm} (7)

where

\[ Q(\xi) = -\frac{1}{2} V_0 c \ a^{-2} \int_0^\infty \left[ f(\xi') - 1 \right] (\xi' - \xi) \ d\xi'. \]  \hspace{1cm} (8)

and

\[ V_0 = \frac{2 a^2 (\rho_p - \rho_f) g}{\mu} \]  \hspace{1cm} (9)

is the settling velocity of an isolated sphere. The matching at \(\xi_0 < \xi \ll b\) yields expressions for the unknowns \(B\) and \(w_\ast\).

The result for the slip velocity is:

\[ w_\ast = Q(0) = -\frac{1}{2} V_0 c \ a^{-2} \int_0^\infty \left[ f(\xi') - 1 \right] \xi' \ d\xi'. \]  \hspace{1cm} (10)

This result is valid for an arbitrary force distribution obeying condition (2) and, more generally, for \(f(\xi)\) that tends to unity sufficiently fast with increasing \(\xi\). In the special case of a step function

\[ f(\xi) = H(\xi - a) \]  \hspace{1cm} (11)

(which is a good approximation for a dilute suspension) we get

\[ w_\ast = \frac{a}{2} V_0 c. \]  \hspace{1cm} (12)

The result (12) was earlier obtained in Ref. 5 in the case the parallel-wall geometry.

**III. CYLINDER WITH A RECTANGULAR CROSS SECTION**

We now consider intrinsic convection in a cylinder with a rectangular cross section, a container well suited for experiments. Let \(b_x\) and \(b_y\) be the horizontal dimensions of the cylinder along directions \(x\) and \(y\), respectively, with

\[ \beta = \frac{b_y}{b_x} < 1. \]  \hspace{1cm} (13)

The outer solution for the vertical component of the suspension velocity can be derived from the classical expression for a Poiseuille flow in a cylinder with a rectangular cross section (see, e.g., Berker):  

\[ w_o = w_\ast + \frac{V}{2} \left[ (Y^2 - \beta^2) + \frac{4}{\beta} \sum_{n=0}^{\infty} (-1)^n \cosh nX \cos nY \right] m^3 \cosh m, \]  \hspace{1cm} (14)

where

\[ X = x/b_x, \ Y = y/b_y, \ m = \left( n + \frac{1}{2} \right) \beta, \]  \hspace{1cm} (15)

and the normalization factor

\[ V = \frac{b_x^2 A}{\mu} \]  \hspace{1cm} (16)

is given by:

\[ V = w_\ast \left[ \frac{\beta^2}{3} + \frac{2}{\beta} \sum_{n=0}^{\infty} (-1)^n \sinh nX \sin nY \right]^{-1}. \]  \hspace{1cm} (17)

The slip velocity \(w_\ast\) can be calculated using Eq. (10) in a general case and is equal to (12) for the model (11).

A characteristic feature of the boundary layer analysis is that it ignores corners. Since the intrinsic convection is driven by the near-wall distribution \(f(\xi)\), it is important to determine whether corners have a significant influence on the suspension flow. To this end, we derived an exact solution of Eqs. (3)–(5) for the present geometry. For simplicity, we considered \(f\) equal to the step function, Eq. (11). The solution is presented in the Appendix and a three dimensional view of the velocity profile is shown in Fig. 1 for a cylinder with aspect ratio \(\beta = 0.8\) and \(\epsilon = 0.1\). A detailed analysis shows that the exact solution differs from the boundary solution by only \(O(\epsilon)\), as expected, and that the effects of corners are limited to regions \(O(\epsilon^2)\).

**IV. PARTICLE DISTRIBUTION NEAR A WALL**

The point-force model with the step-function Stokeslet distribution (11) holds only at small particle volume fractions. Since measuring of small \(O(\epsilon)\) effects is difficult, it is useful to obtain at least an estimate for the intrinsic convection at higher concentrations. As discussed by Bruneau et al.,\(^3\) the near-wall particle distribution is essential in driving the intrinsic convection. We expect that the changes of
the particle-wall distribution with increasing concentration have a dominant effect on the magnitude of the convection.

The equilibrium particle distribution in dense suspension can be evaluated using standard statistical physics methods. It has a well-defined meaning for a system of Brownian particles and for non-Brownian particles can be used to describe a well-mixed state. For hard spheres, the equilibrium particle concentration close to the wall exhibits a pronounced oscillation process. Such correlations result from a complex many-particle evolution process.

For a suspension under shear along a wall, measurements of the particle distribution in the wall region have been reported.\textsuperscript{10} The distribution exhibits an oscillating behavior analogous to that of an equilibrium distribution function,\textsuperscript{7-9} suggesting that excluded volume effects are important also in non-equilibrium states.

In the absence of a better approximation, we use here the equilibrium particle-wall correlation function to estimate the density dependence of the slip velocity. Such a distribution corresponds to an initial stage of sedimentation of a Brownian suspension and a well-mixed suspension of non-Brownian particles. If we assume that at a later time the non-equilibrium particle-wall correlation function still bears some resemblance to the equilibrium structure, we may regard our results as an estimate of a possible magnitude of the concentration effect also at later stages of sedimentation.

Up to the linear order in the volume fraction, the equilibrium particle distribution close to the wall can be determined using standard cluster expansion techniques described, e.g., in Ref. 11. For the particle-wall correlation function \( h(\xi) = f(\xi) - 1 \), where \( \xi = x/a \) is a non-dimensional distance to the wall, one gets:

\[
h(\xi) = h_0(\xi) + c h_1(\xi) + O(c^2),
\]

where

\[
h_0(\xi) = -H(1 - \xi)
\]

and

\[
h_1(\xi) = \frac{1}{4}(\xi^3 - 3 \xi^2 - 9 \xi + 27)H(\xi - 1)(3 - \xi).
\]

Note that \( h_1(1) = 4 \), so that even at volume fractions less than a few percent there is a visible non-uniform structure close to the wall.

At higher volume fractions we have calculated \( h(\xi) \) in the Percus-Yevick (PY) approximation.\textsuperscript{7,8} The expression for \( h(\xi) \) involves the bulk hard-sphere correlation function which for \( \xi \leq 5 \) has been calculated using the analytical solution of the PY equation by Smith and Henderson,\textsuperscript{12} and for \( \xi > 5 \) using the Perram's numerical algorithm.\textsuperscript{13} [Note that there is a misprint in Refs. 7 and 8: the denominator in the expression for \( c_1 \) below Eq. (11) of Ref. 7 should be \((1 - \eta)^4\).]

The equilibrium particle-wall correlation function is plotted in Fig. 2 versus the non-dimensional distance to the wall, at volume fractions \( c = 0.02, 0.1, 0.2, \) and 0.3. We note that even at a low volume fraction \( c = 0.02 \), the particle-wall correlation function differs from its low-density limit (19) in
a visible way. At the volume fraction $c = 0.2$, the density of particles at contact with the wall, $f(1) = 1 + h(1)$, is more than twice as large than in the bulk of the suspension. The particle density assumes values very close to the bulk value at a distance of several particle diameters from the wall. The range of particle-wall correlations increases, however, with the increasing volume fraction.

The correlation function $h(\bar{\xi})$ describes the distribution of the centers of the particles. Experimentally, it is often easier to measure the local volume fraction $\phi(\bar{\xi})$, using, e.g., light-scattering absorption techniques. The distributions $h$ and $\phi$ are related by the formula:

$$\phi(\bar{\xi}) = \frac{3}{4} c \int_{\frac{1}{1+\xi} - 1}^{\xi+1} \left[ 1 - (\bar{\xi} - \eta)^2 \right] f(\eta) \, d\eta. \quad (21)$$

Deviation of the local volume fraction from the bulk value, $\delta\phi(\bar{\xi}) = \phi(\bar{\xi}) - c$, is plotted in Fig. 3.

V. AN ESTIMATE OF CONCENTRATION EFFECTS

At low concentrations, the integral (10) with the particle distribution given by (18)–(20) can be performed analytically. The resulting expression for the slip velocity is:

$$w_* = \frac{9}{4} V_0 c \left[ 1 - \frac{46}{5} c + O(c^2) \right]. \quad (22)$$

Due to the large coefficient ($-46/5$) of the first density correction, even at volume fractions as low as 1% this correction is of the order of 10%.

At higher volume fractions, the slip velocity has been evaluated by using the correlation $h(\bar{\xi})$ calculated in the PY approximation. The results are presented in Fig. 4. We observe that $w_*$ is greatly influenced by the structure of the suspension (assumed to be at equilibrium) close to the wall. In the density range considered, the magnitude of the slip is reduced from the value $w_* = \frac{9}{4} V_0 c$ corresponding to the low-density form of the particle-wall correlation function. Already at a volume fraction as low as $c = 0.02$, the slip velocity is reduced by almost 20%. At a volume fraction $c = 0.2$ the slip is reduced nearly to zero, due to a cancellation of the positive and negative contributions of the oscillating function $h(\bar{\xi})$. At sufficiently low densities the PY result coincides with the analytical result (22). However, a substantial deviation of almost 15% appears already at a density $c = 0.05$.

Due to the factor $\xi'$ of the integrand in Eq. (10), at intermediate and high densities the slip velocity $w_*$ is very sensitive to the behavior of $h(\bar{\xi})$ at $\bar{\xi}$ exceeding 5, where $h(\bar{\xi})$ itself is already small. Due to this “magnification” effect it is difficult to obtain numerically stable results for $c$ greater than 0.2.

VI. CONCLUSION AND DISCUSSION

The model of point forces has been used to describe the three-dimensional convection in a vertical cylinder with a rectangular cross-section, a container suitable for experiments. The exact and the boundary-layer solutions of the intrinsic-convection equations have been derived. It has been concluded that for particles much smaller than the container size the boundary layer analysis is accurate and the effects of the corners are not important.

An estimate of the dependence of the intrinsic convection on the suspension concentration has been obtained using an equilibrium particle distribution in the near wall region. Such distribution corresponds to the initial stages of sedimentation of Brownian suspensions and well-mixed suspensions.
sions of non-Brownian particles. We have found that the oscillations of the distribution yield a substantial reduction of the convection velocity at moderate particle volume fractions. Results for the particle distribution function have been presented for $c$ up to 0.3 and can easily be evaluated also for higher volume fractions. On the other hand, results for the slip velocity have been limited to $c$ up to 0.2 for reason of numerical stability. The model predicts that the intrinsic convection is reduced nearly to zero at a volume fraction $c = 0.2$.

The present model ignores the direct particle-wall and particle-particle hydrodynamic interactions. The hydrodynamic interactions between individual particles and a wall has been considered by two of the authors in a recent paper where the intrinsic convection in a dilute suspension has been analyzed without the point-force simplifying assumption. For the reduced wall slip velocity the result $w_s = 4.395V_0c$ has been obtained, a larger value than $w_s = \frac{1}{2}V_0c$ from the present analysis. An accurate description of the intrinsic convection at moderate and high concentrations would require an analysis of multi-particle hydrodynamic interactions in a presence of a wall. It remains an open question what is the effect of hydrodynamic interactions at such concentrations. On the other hand, one may still expect that the short-range particle order close to the wall results in a substantial reduction of the convection.

Experimental verification of the present model is needed. Measurements of the slip velocity in concentrated suspensions subject to Couette flow has been reported by Jana et al., however, due to a different mechanism of the slip their results cannot be directly compared with ours. Recently, experimental evidence of intrinsic convection has been found by Peysson and Guazzelli. In particular, they have measured convection profiles at different concentrations and compared their results to predictions of the present theory.

APPENDIX: EXACT SOLUTION OF THE GOVERNING EQUATIONS IN A CONTAINER WITH A RECTANGULAR CROSS SECTION

Consider a vertical container with a rectangular cross section. The solution of (3), (5), and (11) for the velocity may be obtained in a form of an expansion in Fourier series, as a function of the pressure gradient. The zero flux condition (4) then determines this pressure gradient. The result is

$$\frac{w}{V_0} = \sum_{n=0}^{\infty} a_n(X) \cos nY,$$

(A1)

where

$$a_n(X) = \begin{cases} a_n^{(1)}(X) & \text{if } |X| \geq 1 - \epsilon, \\ a_n^{(2)}(X) & \text{if } |X| \leq 1 - \epsilon, \end{cases}$$

(A2)

with

$$a_n^{(i)}(X) = \frac{2}{\beta n^3 \cosh n} \left[ A_n(X) + \frac{9c}{2 \epsilon^2} B_n^{(i)}(X) \right],$$

(A3)

in which the $A_n(X)$ and $B_n^{(i)}(X)$ ($i = 1, 2$) functions are the following:

$$A_n(X) = \sin n \beta [ \cosh nX - \cosh n ],$$

$$B_n^{(1)}(X) = -\sin n \beta [ \cosh nX - \cosh n ] + \frac{1}{2} \sin n (\beta - \epsilon) \times [ \cosh(X - \epsilon) - \cosh(X - 2 + \epsilon) ],$$

(A5)

$$B_n^{(2)}(X) = -\sin n \beta m [ \cosh nX - \cosh n ] + \sin n(\beta - \epsilon) \times [ \cosh \epsilon \cosh nX - \cosh n ],$$

(A6)

Applying the zero mean flux condition provides the expression for the pressure gradient:

$$A = -\frac{9 \phi_0}{2 \epsilon^2} \sum_{n=0}^{\infty} \delta_n \int_0^1 B_n^{(1)}(X) \, dX,$$

(A7)

where

$$\delta_n = \frac{(-1)^n}{m^3 \cosh m},$$

(A8)

The integrals in Eq. (A7) can be easily evaluated. They decay like $(-1)^n/n^3$ and $(-1)^n/n^4$ for $n \to \infty$, so that the series in (A7) converge rapidly.